

If the primes are finite, then all of them divide the number one

We propose a novel proof of the infinitude of the primes based on elementary considerations of Legendre's function ϕ , defined in [1, p. 153] as

$$\phi(x, y) = |\{1 \leq n \leq x : \text{integer } n \text{ has no prime factors } \leq y\}|,$$

where x and y are positive integers. The reader can see that

$$\pi(x) = \pi(\sqrt{x}) + \phi(x, \sqrt{x}) - 1,$$

where $\pi(\cdot)$ is the prime-counting function. Let p_1, \dots, p_s be the prime numbers less than or equal to y . Using the inclusion-exclusion principle, it can be proved that

$$\phi(x, y) = x - \sum_{1 \leq i \leq s} \left[\frac{x}{p_i} \right] + \sum_{1 \leq i, j \leq s} \left[\frac{x}{p_i p_j} \right] - \dots + (-1)^s \left[\frac{x}{p_1 \cdots p_s} \right],$$

where $[\cdot]$ is the floor function. Pinasco [2] also used this principle for proving the infinitude of the primes, but his proof is remarkably different from ours.

Suppose that $\{p_1, \dots, p_s\}$ is the set of all prime numbers. Consider $N = p_1 \cdots p_s$. Then $\phi(N^2, N) = 1$. On the other hand, we have

$$\phi(N^2, N) = N^2 - \sum_{1 \leq i \leq s} \left[\frac{N^2}{p_i} \right] + \sum_{1 \leq i, j \leq s} \left[\frac{N^2}{p_i p_j} \right] - \dots + (-1)^s N.$$

Hence, $\phi(N^2, N) = mN$ for some integer m , i.e., every prime number divides 1. This means that all primes, and, consequently, all non-zero natural numbers, are invertible in \mathbb{N} , i.e., we find that \mathbb{N} is a field. This completes the proof.

REFERENCES

1. R. Crandall, C. Pomerance, *Prime numbers. A computational perspective*, Springer-Verlag, New York, 2nd Edition, 2005.
2. J. P. Pinasco, New proofs of Euclid's and Euler's theorems, *Amer. Math. Monthly* **116** (2009) 172–174.

—Submitted by Umberto Cerruti and Nadir Murru

<http://dx.doi.org/10.XXXX/amer.math.monthly.122.XX.XXX>

MSC: Primary 11A41, Secondary 11A99